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Multivariate basic hypergeometric series associated to root systems of type A_m

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Abstract

In previous paper [Ann. Comb. 4 (2000) 347–373], we established and proved two new q-beta integrals and two multivariate basic hypergeometric series associated with the root system of A_m. In this paper we give a detailed proof of the multivariate basic hypergeometric series obtained in [Ann. Comb. 4 (2000) 347–373].

Keywords: q-series; q-beta integrals; Integral transformations; Hypergeometric series very well poised on Lie algebras

1. Introduction and statement of result

In [1] we established and proved two new q-beta integrals associated to the root systems of the classical Lie algebra of type A_m, from which we established the following theorem.

Theorem 1.1. *Let* $m \geqslant 1$ *and* $a, c_1, c_2, c_3, d_1, \ldots, d_m, q \in \mathbb{C}$ *with*

$$\max\{|a|, |c_1|, |c_2|, |c_3|, |d_1|, \ldots, |d_m|, |q|\} < 1.$$

Then we have

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$$\frac{1}{m!(2\pi i)^{m-1}}\int\limits_{T^{m-1}}\frac{\prod\limits_{i\neq j;\,i,j=1}^{m}[z_iz_j^{-1}]_\infty\prod\limits_{j=1}^{m}[Sz_j^{-1}]_\infty}{\prod\limits_{i=1}^{m}\prod\limits_{l=1}^{3}[ac_lz_i]_\infty\prod\limits_{1\leqslant i<j\leqslant m}[az_iz_j]_\infty\prod\limits_{i,j=1}^{m}[d_iz_j^{-1}]_\infty}\frac{dz_1}{z_1}\cdots\frac{dz_{m-1}}{z_{m-1}}$$
$$=\begin{cases}\dfrac{\big[a^{\frac{m}{2}}\big(\prod\limits_{i=1}^{m}d_i\big)\big]_\infty\prod\limits_{k=1}^{m}\big[a^{m+1}\big(\prod\limits_{i=1}^{m}d_i\big)\big(\prod\limits_{l=1}^{3}c_l\big)d_k^{-1}\big]_\infty\prod\limits_{k=1}^{3}\big[a^{\frac{m}{2}+1}\big(\prod\limits_{i=1}^{m}d_i\big)\big(\prod\limits_{l=1}^{3}c_l\big)c_k^{-1}\big]_\infty}{[q]_\infty^{m-1}[a^{\frac{m}{2}}]_\infty\big[\prod\limits_{i=1}^{m}d_i\big]_\infty\prod\limits_{i=1}^{m}\prod\limits_{j=1}^{3}[ad_ic_j]_\infty\prod\limits_{1\leqslant i<j\leqslant 3}[a^{\frac{m}{2}+1}c_ic_j]_\infty\prod\limits_{1\leqslant i<j\leqslant m}[ad_id_j]_\infty}, \\ \quad m \text{ is even}, \\ \dfrac{\big[a^{\frac{m+3}{2}}\big(\prod\limits_{l=1}^{3}c_l\big)\big(\prod\limits_{i=1}^{m}d_i\big)\big]_\infty\prod\limits_{k=1}^{m}\big[a^{m+1}\big(\prod\limits_{l=1}^{3}c_l\big)\big(\prod\limits_{i=1}^{m}d_i\big)d_k^{-1}\big]_\infty\prod\limits_{l=1}^{3}\big[a^{\frac{m+1}{2}}\big(\prod\limits_{i=1}^{m}d_i\big)c_l\big]_\infty}{[q]_\infty^{m-1}\big[\prod\limits_{i=1}^{m}d_i\big]_\infty\big[a^{\frac{m+3}{2}}\prod\limits_{l=1}^{3}c_l\big]_\infty\prod\limits_{l=1}^{3}[a^{\frac{m+1}{2}}c_l]_\infty\prod\limits_{l=1}^{3}\prod\limits_{k=1}^{m}[ac_ld_k]_\infty\prod\limits_{1\leqslant i<j\leqslant m}[ad_id_j]_\infty}, \\ \quad m \text{ is odd},\end{cases} \tag{1.1}$$

where $S=a^{m+1}(\prod_{l=1}^{3}c_l)(\prod_{i=1}^{m}d_i)$ and $\prod_{j=1}^{m}z_j=1$ and the integral in each variable $z_1,\dots,z_{m-1}$ is over the unit circle T taken in the positive direction.

In the case of $m=1$, integral (1.1) reduces to the Askey–Wilson integral [4], and in case of $m=2$, it reduces to the integral in [3, Theorem 3.1]. In conjunction with these integrals, we also defined two new kinds of basic hypergeometric series associated to the root system A_m.

Theorem 1.2. *With notation as in Theorem* 1.1 *and $\prod_{i=1}^{m}d_i=q^{-k}$, we will have*

$$\sum_{x_1+\cdots+x_m=k}\frac{\prod\limits_{i=1}^{m}\prod\limits_{j=1}^{3}[ac_jd_i]_{x_i}\prod\limits_{1\leqslant i<j\leqslant m}[ad_id_j]_{x_i+x_j}\prod\limits_{i,j=1}^{m}[d_id_j^{-1}]_{-x_j}}{\prod\limits_{i\neq j;\,i,j=1}^{m}[d_id_j^{-1}]_{x_i-x_j}\prod\limits_{j=1}^{m}\big[a^{m+1}\big(\prod\limits_{i=1}^{m}d_i\big)\big(\prod\limits_{k=1}^{3}c_k\big)d_j^{-1}\big]_{-x_j}}$$
$$=\begin{cases}\dfrac{[1]_{-k}}{[a^{\frac{m}{2}}]_{-k}\prod\limits_{1\leqslant i<j\leqslant 3}[a^{\frac{m+2}{2}}c_ic_j]_{-k}}, & m \text{ is even},\\ \dfrac{[1]_{-k}}{\prod\limits_{i=1}^{3}[a^{\frac{m-1}{2}}c_i]_{-k}\big[a^{\frac{m+3}{2}}\prod\limits_{i=1}^{3}c_i\big]_{-k}}, & m \text{ is odd}.\end{cases} \tag{1.2}$$

One application of the integrals and the series identities, such as Theorems 1.1 and 1.2, would be in the theory of orthogonal polynomials of several variables. It is possible that they will be useful in finding the generating functions and the reproducing kernels for some of Macdonald's polynomials [3].

An interesting special case of (1.2) was obtained in [2], in which we proved that when $m=2n$, $n=1$ the series is equivalent to Jackson's ${}_8\Psi_7$ sum.

In this paper we will give a detailed prove of Theorem 1.2. We will discuss the proof for the even case and the odd one in two different sections. To start with, we need the following set of identities:

$$\big[a_ja_i^{-1}q^{-k}\big]_\infty=\frac{[a_ja_i^{-1}]_\infty}{[a_ja_i^{-1}]_{-k}}, \tag{1.3}$$

$$\frac{[aq^{-m}]_\infty}{[a]_\infty} = [aq^{-m}]_m, \tag{1.4}$$

$$[a]_{-m} = \frac{1}{[aq^{-m}]_m}. \tag{1.5}$$

2. Proof of Theorem 1.2

We will prove Theorem 1.2 by expanding the integral (1.1) as a multiple series of residues and then making the series terminate. We will first make repeated use of Cauchy's theorem to show that the integral (1.1) is equal to the series of residues. We will then discuss the convergence of the series of residues for the integral (1.1). Finally, we will show how terminating the series will allows us to drop various terms.

Let us assume that we evaluate the multiple integrals in (1.1) in the order of z_1 first, and then $z_2, \ldots, z_{m-1}$, where $z_m = (z_1 \ldots z_{m-1})^{-1}$. There are two broad types of poles corresponding to the variable z_i for $1 \leqslant i \leqslant m-1$. The first type are poles of the integrand in (1.1) corresponding to factors in the denominator of the integrand in (1.1) which contain the variable z_i directly, for example, the factors $(1 - d_2 q^4 z_i^{-1})$ or $(1 - aq^4 z_i z_j)$, $j \neq m$. The second type are poles of the integrand in (1.1) corresponding to factors of the denominator which contain the variable $z_m = (z_1 \ldots z_i \ldots z_{m-1})^{-1}$, for example, the factors $(1 - ac_2 q^4 z_m) = (1 - ac_2 q^4 z_1^{-1} \ldots z_{m-1}^{-1})$ or $(1 - aq^4 z_m z_j)$, $j \neq i$. It will be convenient to view poles of the second type as poles with respect to the variable z_m with suitable care for the sign of the residue. This will also interchange the role of z_i and z_m for the remaining variables of integration, i.e., the value of z_m will be fixed and $z_i = (z_1 \ldots z_{i-1} z_{i+1} \ldots z_m)^{-1}$ is variable. It will be thus simpler, when we have a residue of a pole of z_i of the second type, to interchange the variables z_i and z_m so that z_i will be fixed and z_m be variable. We will also need to multiply the corresponding residue by -1 as will be seen below.

2.1. Convergence of the series of residues

With notations as in integral (1.1), and $0 \leqslant k \leqslant m-1$, let

$$I_k = \{1, \ldots, k\},$$
$$A = \{d_1, d_2, \ldots, d_m\},$$
$$B^{-1} = \bigl\{(ac_1)^{-1}, (ac_2)^{-1}, (ac_3)^{-1}\bigr\}.$$

Let $\Delta(k)$ be the set of all $\delta(k) = (\delta_1, \ldots, \delta_m)$ such that $\delta_i = (\gamma_i, \sigma_i)$ for $1 \leqslant i \leqslant m$, where

$$L \subseteq I_k, \quad M = I_k - L \quad \text{are two sets of indices and}$$
$$\gamma_i \in A \cup B^{-1} \quad \text{for } i \in L,$$
$$\gamma_i = (a\gamma_j)^{-1}, \quad j \in L, \quad \text{or} \quad \gamma_i = (az_l)^{-1}, \quad m > l > k, \ \text{for } i \in M,$$
$$\gamma_i = z_i \quad \text{for } k < i \leqslant m-1,$$

$$\gamma_m = \prod_{i=1}^{m-1} \gamma_i^{-1},$$
$$\gamma_j \neq \gamma_l \quad \text{for } 1 \leqslant j \neq l \leqslant m,$$
$$\sigma_i = \begin{cases} 1 & \text{for } k < i \leqslant m, \\ \pm 1 & \text{otherwise,} \end{cases}$$

where σ_i represents the sign -1 or $+1$ according to the different type of the poles in the evaluation of the integral (1.1).

- When $\sigma_i = 1$, γ_i corresponds to a pole of the integral
$$\frac{1}{2\pi i} \int_T \frac{1}{(1 - Az_i^{-1})} \frac{dz_i}{z_i}, \quad |A| < 1,$$
that is, the residue of $1/((1 - Az_i^{-1})z_i) = 1$ at $z_i = A$. This also includes the case of
$$\frac{1}{2\pi i} \int_T \frac{1}{(1 - Bz_m)} \frac{dz_i}{z_i}, \quad B \in \mathbb{C}.$$
- When $\sigma_i = -1$, γ_i corresponds to a pole of the integral
$$\frac{1}{2\pi i} \int_T \frac{1}{(1 - Az_i)} \frac{dz_i}{z_i} = \frac{1}{2\pi i} \int_T \frac{1}{A} \frac{1}{(A^{-1} - z_i)} \frac{dz_i}{z_i}, \quad |A| > 1,$$
that is, the residue of $1/((1 - Az_i)z_i) = -1$ at $z_i = A^{-1}$. This also includes the case of
$$\frac{1}{2\pi i} \int_T \frac{1}{(1 - Bz_m^{-1})} \frac{dz_i}{z_i}, \quad B \in \mathbb{C}.$$

Denote $\Delta(m-1)$ by Δ and $\delta(m-1)$ by δ. Let T^ε be the circle of radius ε centered at zero and traversed in the positive direction. We can choose the radius ε so that the denominator of the integrand in (1.1) does not vanish for $|z_i| = \varepsilon$ or $|z_i| = 1$ for $1 \leqslant i \leqslant m-1$. Similarly, we will later choose ε so that the circle T^ε does not pass through the poles of other integrands we will consider.

For $0 \leqslant k \leqslant m-1$, $\delta(k) \in \Delta(k)$ and $\varepsilon \geqslant 0$, let $A_{\delta,\varepsilon}(k)$ be the set of all $(y_1, \ldots, y_m) \in Z^n$ such that $z_1 = \gamma_1 q^{y_1}, \ldots, z_k = \gamma_k q^{y_k}$ is a sequence of poles of the integrand (1.1) with respect to the variables $z_1, \ldots, z_k$; $y_{k+1}, \ldots, y_{m-1} = 0$; $y_m = -(y_1 + \cdots + y_{m-1})$; and with the further assumption that for each i, $1 \leqslant i \leqslant k$, either

$$\varepsilon < |\gamma_k q^{y_k}| < 1$$

or

$$\varepsilon < \left| \prod_{j=1}^{k} \gamma_j^{-1} q^{-y_k} \right| < 1.$$

For $0 \leqslant k \leqslant m-1$, $\delta(k) \in \Delta(k)$ and $\varepsilon \geqslant 0$, we shall consider

$$F_{\delta,\varepsilon}(k)=\sum_{(y_1,\dots,y_m)\in A_{\delta,\varepsilon}(k)}\frac{\left(\prod_{i=1}^{m-1}\sigma_i\right)\prod_{i\neq j;\,i,j=1}^{m}[\gamma_i\gamma_j^{-1}q^{y_i-y_j}]_\infty\prod_{j=1}^{m}[S\gamma_j^{-1}q^{-y_j}]_\infty}{\prod'_{1\leqslant i\leqslant m}\prod_{l=1}^{3}[ac_l\gamma_iq^{y_i}]_\infty\prod'_{1\leqslant i<j\leqslant m}[a\gamma_i\gamma_jq^{y_i+y_j}]_\infty\prod'_{1\leqslant i,j\leqslant m}[d_i\gamma_j^{-1}q^{-y_j}]_\infty},\tag{2.1}$$

where $\prod'$ means the usual product except that if $c=q^{-l}$ for some nonnegative integer l, then the factor $[c]_\infty$ in the product is replaced by $[q^{-l}]_l[q]_\infty$. Also, we shall denote $a^{m+1}(\prod_{l=1}^{3}c_l)(\prod_{i=1}^{m}d_i)$ by S. We will denote $\lim_{\varepsilon\to0}F_{\delta,\varepsilon}(k)$ by $F_\delta(k)$.

We can write the sum (2.1) as follows:

$$F_{\delta,\varepsilon}(k)=A\sum_{(y_1,\dots,y_m)\in A_{\delta,\varepsilon}(k)}\frac{\prod_{i=1}^{m\prime}\prod_{l=1}^{3}[ac_l\gamma_i]_{y_i}\prod'_{1\leqslant i<j\leqslant m}[a\gamma_i\gamma_j]_{y_i+y_j}\prod_{i,j=1}^{m\prime}[d_i\gamma_j^{-1}]_{-y_j}}{\prod_{i\neq j;\,i,j=1}^{m}[\gamma_i\gamma_j^{-1}]_{y_i-y_j}\prod_{j=1}^{m}[S\gamma_j^{-1}]_{-y_j}},\tag{2.2}$$

where

$$A=\frac{\left(\prod_{i=1}^{m-1}\sigma_i\right)\prod_{i\neq j;\,i,j=1}^{m}[\gamma_i\gamma_j^{-1}]_\infty\prod_{j=1}^{m}[S\gamma_j^{-1}]_\infty}{\prod'_{1\leqslant i\leqslant m}\prod_{l=1}^{3}[ac_l\gamma_i]_\infty\prod'_{1\leqslant i<j\leqslant m}[a\gamma_i\gamma_j]_\infty\prod'_{1\leqslant i,j\leqslant m}[d_i\gamma_j^{-1}]_\infty}.$$

Expression (2.2) can be rewritten as

$$F_{\delta,\varepsilon}(k)=A\sum_{(y_1,\dots,y_m)\in A_{\delta,\varepsilon}(k)}\frac{\prod_{i=1}^{m}\prod_{l=1}^{3}[ac_l\gamma_i]_{y_i}\prod_{1\leqslant i<j\leqslant m}[a\gamma_i\gamma_j]_{y_i+y_j}\sum_{\rho\in S_m}\operatorname{sgn}(\rho)\prod_{i=1}^{m}(\gamma_{\rho(i)}q^{y_{\rho(i)}})^{m-i}}{\prod_{i,j=1}^{m}[\frac{q}{d_i\gamma_j^{-1}}]_{y_j}\prod_{1\leqslant i<j\leqslant m}(\gamma_i-\gamma_j)\prod_{j=1}^{m}[S\gamma_j^{-1}]_{y_1+\dots+\hat{y}_j+\dots+y_m}},\tag{2.3}$$

where $\operatorname{sgn}(\rho)$, is the sign of the permutation $\rho\in S_m$ the symmetric group on m letters.

To show the absolute convergence of the series in Eq. (2.3), it will be easier to consider a related series whose terms contain the terms in (2.3) as a subset. Writing this new series in a form to show each terms dependence on the y_1 summation index, we will consider the absolute convergence of the following series:

$$\begin{aligned}&A\sum_{\rho\in S_m}\sum_{y_{m-1}=-\infty}^{\infty}\dots\sum_{y_2=-\infty}^{\infty}\sum_{y_1=-\infty}^{\infty}\frac{\prod_{i=1}^{m-1}\prod_{l=1}^{3}[ac_l\gamma_i]_{y_i}\prod_{l=1}^{3}[ac_l\gamma_mq^{-y_2-\dots-y_{m-1}}]_{-y_1}[ac_l\gamma_m]_{-y_2-\dots-y_{m-1}}}{\prod_{i=1}^{m}\prod_{j=1}^{m-1}[\frac{q}{d_i}\gamma_j]_{y_j}\prod_{i=1}^{m}[\frac{q}{d_i}\gamma_mq^{-y_2-\dots-y_{m-1}}]_{-y_1}[\frac{q}{d_i}\gamma_m]_{-y_2-\dots-y_{m-1}}}\\&\quad\times\frac{\prod_{1\leqslant i<j\leqslant m-1}[a\gamma_i\gamma_j]_{y_i+y_j}\prod_{i=1}^{m-1}[a\gamma_i\gamma_mq^{-y_2-\dots-\hat{y}_i-\dots-y_{m-1}}]_{-y_1}[a\gamma_i\gamma_m]_{-y_2-\dots-\hat{y}_i-\dots-y_{m-1}}}{\prod_{1\leqslant i<j\leqslant m}(\gamma_i-\gamma_j)}\\&\quad\times\frac{[a\gamma_1\gamma_m]_{-y_2-\dots-y_{m-1}}\operatorname{sgn}(\rho)\prod_{i=1}^{m}(\gamma_iq^{y_i})^{m-\rho(i)}}{\prod_{j=1}^{m-1}[S\gamma_j^{-1}]_{-y_j}[S\gamma_m^{-1}q^{y_2+\dots+y_{m-1}}]_{y_1}[S\gamma_m^{-1}]_{y_2+\dots+y_{m-1}}}.\end{aligned}\tag{2.4}$$

To prove the absolute convergence for series (2.4), we will apply induction on the sum over the y indices. We will prove that for fixed $y_2,\dots,y_{m-1}$ the sum in (2.4) over $-\infty<y_1<\infty$ converges absolutely. Then will assume that for fixed $y_{k+1},\dots,y_{m-1}$ the sum in

(2.4) over $-\infty < y_1 < \infty, \ldots, -\infty < y_k < \infty$ converges absolutely. We will show that for fixed $y_{k+2}, \ldots, y_{m-1}$ the sum in (2.4) over $-\infty < y_1 < \infty, \ldots, -\infty < y_{k+1} < \infty$ converges absolutely.

In the first case, just the sum in (2.4) over the y_1 index, we will rewrite expression (2.4) as

$$A \sum_{\rho \in S_m} \sum_{y_{m-1}} \ldots \sum_{y_2} \left\{ B_2 \sum_{y_1} C_1(y_1) \right\}, \tag{2.5}$$

where $C_1(y_1)$ represents the factors involving the term y_1 of the series (2.4), which can be written as follows:

$$C_1(y_1) = \frac{\prod_{l=1}^{3} [ac_l\gamma_1]_{y_1} \prod_{l=1}^{3} [ac_l\gamma_m q^{-y_2-\cdots-y_{m-1}}]_{-y_1} \prod_{i=2}^{m-1} [a\gamma_1\gamma_i q^{y_i}]_{y_1}}{\prod_{i=1}^{m} [\frac{q}{d_i}\gamma_1]_{y_1} \prod_{i=1}^{m} [\frac{q}{d_i}\gamma_m q^{-(y_2+\cdots+y_{m-1})}]_{-y_1}}
\times \frac{\prod_{i=2}^{m-1} [a\gamma_i\gamma_m q^{-(y_2+\cdots+\hat{y}_i+\cdots+y_{m-1})}]_{-y_1} \operatorname{sgn}(\rho)(\gamma_1 q^{y_1})^{m-\rho(1)}}{[S\gamma_1^{-1}]_{-y_1} [S\gamma_m^{-1} q^{(y_2+\cdots+y_{m-1})}]_{y_1}}$$

and B_2 represents the factors involving the remaining terms $y_2, \ldots, y_{m-1}$, which can be written as follows:

$$B_2 = \frac{\prod_{i=2}^{m-1} \prod_{l=1}^{3} [ac_l\gamma_i]_{y_i} [ac_l\gamma_m]_{-y_2-\cdots-y_{m-1}} \prod_{i=2}^{m} [a\gamma_1\gamma_i]_{y_i} \operatorname{sgn}(\rho) \prod_{i=2}^{m} (\gamma_i q^{y_i})^{m-\rho(i)}}{\prod_{i=1}^{m} \prod_{j=2}^{m-1} [\frac{q}{d_i}\gamma_j]_{y_j} [\frac{q}{d_i}\gamma_m]_{-y_2-\cdots-y_{m-1}} \prod_{2 \leqslant i < j \leqslant m} (\gamma_i - \gamma_j) \prod_{j=2}^{m-1} [S\gamma_j^{-1}]_{-y_j} [S\gamma_m^{-1}]_{y_2+\cdots+y_{m-1}}}
\times \prod_{2 \leqslant i < j \leqslant m-1} [a\gamma_i\gamma_j]_{y_i+y_j} [a\gamma_i\gamma_m]_{-y_2-\cdots-\hat{y}_i-\cdots-y_{m-1}}.$$

We want to consider the absolute convergence of

$$\sum_{-\infty < y_1 < \infty} C_1(y_1) = \sum_{y_1=0}^{\infty} C_1(y_1) + \sum_{-\infty < y_1 < 0} C_1(y_1).$$

We will first apply the ratio test to $\sum_{y_1=0}^{\infty} C_1(y_1)$, that is $\lim_{y_1 \to \infty} \left| \frac{C_1(y_1+1)}{C_1(y_1)} \right|$, and since $|q| < 1$, as $y_1 \to \infty$, q^{y_1} always goes to zero, then

$$\lim_{y_1 \to \infty} \left| \frac{C_1(y_1+1)}{C_1(y_1)} \right| = \frac{S\gamma_m^m q^m q^{m-\rho(1)} \gamma_1^{-1}}{\left(\prod_{i=1}^{m} d_i\right)(a^3 c_1 c_2 c_3 \gamma_m^3)\left(a^{m-2}\left(\prod_{i=2}^{m-1} \gamma_i\right)\gamma_m^{m-2}\right)}
\times \frac{q^{-m(y_2+\cdots+y_{m-1})}}{q^{-3(y_2+\cdots+y_{m-1})-(m-3)(y_2+\cdots+y_{m-1})}}
= |q^{2m-\rho(1)}| < 1, \quad \rho(1) \leqslant m,$$

that is

$$|q^{2m-\rho(1)}| < |q^m| < 1,$$

where $\gamma_1 \ldots \gamma_m = 1$. Hence $\sum_{y_1=0}^{\infty} C_1(y_1)$ is absolutely convergent.

In order to apply the ratio test to $\sum_{-\infty<y_1<0} C_1(y_1)$ we will have

$$\lim_{y_1\to-\infty}\left|\frac{C_1(y_1)}{C_1(y_1+1)}\right| = \lim_{y_1\to-\infty}\left|\frac{\prod_{i=1}^{m}(1-\frac{q}{d_i}\gamma_1 q^{y_1})(1-S\gamma_m^{-1}q^{(y_1+y_2+\cdots+y_{m-1})})q^{-m+\rho(1)}}{\prod_{l=1}^{3}(1-ac_l\gamma_1 q^{y_1})\prod_{i=2}^{m-1}(1-a\gamma_1\gamma_i q^{y_i+y_1})}\right|$$
$$= |q^{\rho(1)}|, \quad \text{where } \rho(1)\geqslant 1,$$

that is

$$|q^{\rho(1)}| < |q| < 1,$$

which means that $\sum_{-\infty<y_1<0} C_1(y_1)$ and hence $\sum_{-\infty<y_1<\infty} C_1(y_1)$ is absolutely convergent.

Now assume that for fixed $y_{k+1},\dots,y_{m-1}$ the sum in (2.4) over $-\infty<y_1<\infty,\dots,$ $-\infty<y_k<\infty$ converges absolutely. We will show that for fixed $y_{k+2},\dots,y_{m-1}$ the sum in (2.4) over $-\infty<y_1<\infty,\dots,-\infty<y_{k+1}<\infty$ converges absolutely.

We can write (2.4) as follows:

$$A\sum_{\rho\in S_m}\sum_{y_{m-1}}\dots\sum_{y_{k+2}}\left\{B_{k+2}\sum_{y_{k+1}}C_{k+1}(y_{k+1})\sum_{y_k}C_k(y_k)\dots\sum_{y_1}C_1(y_1)\right\}, \tag{2.6}$$

where $C_{k+1}(y_{k+1})$ represents the factors involving the term y_{k+1} (and not involving $y_1,\dots,y_k$) of the series (2.4), which can be written as follows:

$$C_{k+1}(y_{k+1}) = \frac{\prod_{l=1}^{3}[ac_l\gamma_{k+1}]_{y_{k+1}}\prod_{l=1}^{3}[ac_l\gamma_m q^{-y_{k+2}-\cdots-y_{m-1}}]_{-y_{k+1}}\prod_{i=k+2}^{m-1}[a\gamma_{k+1}\gamma_i q^{y_i}]_{y_{k+1}}\prod_{i=1}^{k}[a\gamma_{k+1}\gamma_i]_{y_{k+1}}}{\prod_{i=1}^{m}[\frac{q}{d_i}\gamma_{k+1}]_{y_{k+1}}\prod_{i=1}^{m}[\frac{q}{d_i}\gamma_m q^{-(y_{k+2}+\cdots+y_{m-1})}]_{-y_{k+1}}}$$
$$\times\frac{\prod_{i=k+2}^{m-1}[a\gamma_i\gamma_m q^{-(y_{k+2}+\cdots+\hat{y}_i+\cdots+y_{m-1})}]_{-y_{k+1}}\prod_{i=1}^{k}[a\gamma_i\gamma_m q^{-(y_{k+2}+\cdots+y_{m-1})}]_{-y_{k+1}}}{[S\gamma_{k+1}^{-1}]_{-y_{k+1}}[S\gamma_m^{-1}q^{(y_{k+2}+\cdots+y_{m-1})}]_{y_{k+1}}}$$
$$\times\operatorname{sgn}(\rho)(\gamma_{k+1}q^{y_{k+1}})^{m-\rho(k+1)}$$

and B_{k+2} represents the factors involving terms $y_{k+2},\dots,y_{m-1}$, which can be written as follows:

$$B_{k+2} = \frac{\prod_{i=k+2}^{m-1}\prod_{l=1}^{3}[ac_l\gamma_i]_{y_i}[ac_l\gamma_m]_{-y_{k+2}-\cdots-y_{m-1}}}{\prod_{i=1}^{m}\prod_{j=k+2}^{m-1}[\frac{q}{d_i}\gamma_j]_{y_j}[\frac{q}{d_i}\gamma_m]_{-y_{k+2}-\cdots-y_{m-1}}\prod_{k+2\leqslant i<j\leqslant m}(\gamma_i-\gamma_j)}$$
$$\times\frac{\prod_{k+2\leqslant i<j\leqslant m-1}[a\gamma_i\gamma_j]_{y_i+y_j}[a\gamma_i\gamma_m]_{-y_{k+2}-\cdots-\hat{y}_i-\cdots-y_{m-1}}}{\prod_{j=k+2}^{m-1}[S\gamma_j^{-1}]_{-y_j}[S\gamma_m^{-1}]_{y_{k+2}+\cdots+y_{m-1}}}\operatorname{sgn}(\rho)\prod_{i=k+2}^{m}(\gamma_i q^{y_i})^{m-\rho(i)}.$$

We want to consider the absolute convergence of

$$\sum_{-\infty<y_{k+1}<\infty}C_{k+1}(y_{k+1}) = \sum_{y_{k+1}=0}^{\infty}C_{k+1}(y_{k+1}) + \sum_{-\infty<y_{k+1}<0}C_{k+1}(y_{k+1}).$$

We will first apply the ratio test to $\sum_{y_{k+1}=0}^{\infty} C_{k+1}(y_{k+1})$, that is $\lim_{y_{k+1}\to\infty} \left|\frac{C_{k+1}(y_{k+1}+1)}{C_{k+1}(y_{k+1})}\right|$. Since $|q| < 1$, as $y_{k+1} \to \infty$, $q^{y_{k+1}}$ always goes to zero, then

$$\lim_{y_{k+1}\to\infty} \left|\frac{C_{k+1}(y_{k+1}+1)}{C_{k+1}(y_{k+1})}\right| = |q^{2m-\rho(k+1)}| < 1, \quad 1 \leqslant \rho(k+1) = m,$$

that is

$$|q^{2m-\rho(k+1)}| < |q^m| < 1,$$

where $\gamma_1 \ldots \gamma_m = 1$, that means that $\sum_{y_{k+1}=0}^{\infty} C_{k+1}(y_{k+1})$, is absolutely convergent.

In order to apply the ratio test to $\sum_{-\infty<y_{K+1}<0} C_{k+1}(y_{k+1})$, then we will have

$$\begin{aligned}\lim_{y_{k+1}\to-\infty} \left|\frac{C_{k+1}(y_{k+1})}{C_{k+1}(y_{k+1}+1)}\right| &= |q^m q^{-m+\rho(k+1)}| \\ &= |q^{\rho(k+1)}|, \quad \text{where } \rho(k+1) \geqslant 1,\end{aligned}$$

that is

$$|q^{\rho(k+1)}| < |q| < 1,$$

which means that $\sum_{y_{k+1}=-\infty}^{\infty} C_{k+1}(y_{k+1})$, is also absolutely convergent. By induction on the indices y it follows that the series (2.4) is absolutely convergent.

As a consequence, we have the following

Lemma. *With notation as above and for* $0 \leqslant k \leqslant m-1$, *we have*

$$\begin{aligned}&\frac{1}{(2\pi i)^{m-1}} \int\limits_{T^{m-1}} \frac{\prod_{i\neq j;\, i,j=1}^{m} [z_i z_j^{-1}]_\infty \prod_{j=1}^{m} [S z_j^{-1}]_\infty}{\prod_{i=1}^{m}\prod_{l=1}^{3} [ac_l z_i]_\infty \prod_{1\leqslant i<j\leqslant m} [a z_i z_j]_\infty \prod_{i,j=1}^{m} [d_i z_j^{-1}]_\infty} \frac{dz_1}{z_1} \cdots \frac{dz_{m-1}}{z_{m-1}} \\ &\quad = \frac{1}{(2\pi i)^{m-1-k}} \int\limits_{T^{m-1-k}} \sum_{\delta(k)\in\Delta(k)} F_\delta(k) \frac{dz_{k+1}}{z_{k+1}} \cdots \frac{dz_{m-1}}{z_{m-1}}.\end{aligned} \tag{2.7}$$

Proof. The identity (2.7) is trivially true for $k = 0$. Suppose identity (2.4) is true for $0 \leqslant k = l \leqslant m-1$. We will then show it is true for $k = l+1$. Applying Cauchy's theorem, it follows that

$$\begin{aligned}&\frac{1}{(2\pi i)^{m-1-l}} \int\limits_{T^{m-1-l}} \sum_{\delta(l)\in\Delta(l)} F_\delta(l) \frac{dz_{l+1}}{z_{l+1}} \cdots \frac{dz_{m-1}}{z_{m-1}} \\ &\quad = \frac{1}{(2\pi i)^{m-1-l}} \int\limits_{T^{m-2-l}} \int\limits_{T^{\varepsilon}} \sum_{\delta(l)\in\Delta(l)} F_\delta(l) \frac{dz_{l+1}}{z_{l+1}} \cdots \frac{dz_{m-1}}{z_{m-1}} \\ &\qquad + \frac{1}{(2\pi i)^{m-2-l}} \int\limits_{T^{m-2-l}} \sum_{\delta(l+1)\in\Delta(l+1)} F_{\delta,\varepsilon}(l+1) \frac{dz_{l+2}}{z_{l+2}} \cdots \frac{dz_{m-1}}{z_{m-1}}.\end{aligned}$$

Now

$$\int\limits_{T^{\varepsilon}} \sum_{\delta(l)\in\Delta(l)} F_\delta(l) \frac{dz_{l+1}}{z_{l+1}} \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.8}$$

This is because for $\delta(l) \in \Delta(l)$, $|z_{l+1}| = \varepsilon$, $|z_i| = 1$, $l+1 < i < 2n$, the series $F_\delta(l)$ is bounded in absolute value by a series of the form

$$\begin{aligned}
&D \sum_{\rho\in S_m} \sum_{y_l=-\infty}^{\infty} \cdots \sum_{y_2=-\infty}^{\infty} \sum_{y_1=-\infty}^{\infty} \left| \frac{\prod_{i=1}^{m-1}\prod_{l=1}^{3}[ac_l u_i]_{y_i} \prod_{l=1}^{3}[ac_l u_m q^{-y_2-\cdots-y_{m-1}}]_{-y_1}[ac_l u_m]_{-y_2-\cdots-y_{m-1}}}{\prod_{i=1}^{m}\prod_{j=1}^{m-1}[\frac{q}{d_i}u_j]_{y_j} \prod_{i=1}^{m}[\frac{q}{d_i}u_m q^{-y_2-\cdots-y_{m-1}}]_{-y_1}[\frac{q}{d_i}u_m]_{-y_2-\cdots-y_{m-1}}} \right| \\
&\quad\times \Bigg| \prod_{1\leqslant i<j\leqslant m-1} [au_iu_j]_{y_i+y_j} \\
&\quad\times \prod_{i=1}^{m-1} [au_iu_m q^{-y_2-\cdots-\hat{y}_i-\cdots-y_{m-1}}]_{-y_1}[au_iu_m]_{-y_2-\cdots-\hat{y}_i-\cdots-y_{m-1}} \Bigg| \\
&\quad\times \left| \frac{[au_1u_m]_{-y_2-\cdots-y_{m-1}} \prod_{i=1}^{m}(u_iq^{y_i})^{m-\rho(i)}}{\prod_{j=1}^{m-1}[Su_j^{-1}]_{-y_j}[Su_m^{-1}q^{y_2+\cdots+y_{m-1}}]_{y_1}[Su_m^{-1}]_{y_2+\cdots+y_{m-1}}} \right|,
\end{aligned} \tag{2.9}$$

where $y_m = -(y_1 + \cdots + y_{m-1})$, $D > 0$ is a constant, $u_i = \gamma_i$ for $1 \leqslant i \leqslant l$, $|q| < |u_{l+1}| \leqslant 1$, $u_{l+1}q^{y_{l+1}} = z_{l+1}$ with $z_{l+1} = \varepsilon$, $u_i = z_i$ for $l+1 \leqslant i \leqslant m$, and $\prod_{i=1}^{m} z_i = 1$. Of course this is essentially the same series as in (2.4). As a consequence of the argument immediately above, the series (2.8) converges absolutely and uniformly in z_i for $l+1 \leqslant i < m$. Letting $\varepsilon = \varepsilon_0|q|^{y_{l+1}}$ for some $|q| < \varepsilon_0 \leqslant 1$, it follows that $F_\delta(l) \to 0$ as $y_{l+1} \to \infty$. Hence the $k = l+1$ case (and by induction the general case) of identity (2.7) follows. □

2.2. *Poles of the integrand in (1.1)*

We will further consider the series expansion on the right-hand side of identity (2.7) in the case $k = m-1$. In this case the integral on the r.h.s. of (2.7) disappears and we are left with simply series $F_\delta(m-1)$, which is the $k = m-1$, $\varepsilon = 0$ case of the series in (2.1). The series can also be viewed as the multiple residue expansion for the integral in (1.1).

We will multiply both sides of the integral (1.1) by $[d_1 \ldots d_m]_\infty$, and we will determine which terms in the series $F_\delta(m-1)$ remain after setting $d_1 \ldots d_m = q^{-k}$. In other words, we are looking for these terms in $F_\delta(m-1)$ which contain the factor $1 - q^k d_1 \ldots d_m$, in their denominator.

Let us consider the kinds of possible poles with respect the variables of integration of the integral (1.1) in the order z_1 first. We find the following possible poles from each kind different factor in the denominator of the integrand:

(1) From the factor $\prod_{i=1}^{m}\prod_{j=1}^{3}[ac_jz_i]_\infty$ in (1.1), we have the possible poles $\gamma_i = (ac_j)^{-1}q^{-y_i}$, $1 \leqslant j \leqslant 3$ and $1 \leqslant i \leqslant m-1$.
(2) From the factor $\prod_{i,j=1}^{m}[d_jz_i^{-1}]_\infty$ in (1.1), we have the possible poles $\gamma_i = d_jq^{-y_i}$, $1 \leqslant j \leqslant m$ and $1 \leqslant i \leqslant m-1$.
(3) From the factor $\prod_{1\leqslant i<j\leqslant m}[az_iz_jq^{y_i+y_j}]_\infty$ in (1.1), we have the possible poles $\gamma_i = a^{-1}z_j^{-1}q^{y_i}$ that is:
    (a) $\gamma_i = a^{-1}((ac_j)^{-1})^{-1}q^{-y_i} = c_jq^{-y_i}$, $1 \leqslant j \leqslant 3$ and $1 \leqslant i \leqslant m-1$;

(b) $\gamma_i = a^{-1}(d_j)^{-1}q^{-y_i} = a^{-1}d_j^{-1}q^{-y_i}$, $1 \leqslant i \leqslant m-1$;
(c) $\gamma_i = a^{-1}\gamma_k^{-1}q^{-y_i}$, $1 \leqslant i, k$; $i \neq k \leqslant m-1$;
(d) $\gamma_i = \gamma_1^{-1}\gamma_2^{-1}\dots\hat{\gamma}_i^{-1}\dots\gamma_m^{-1}$, where the remaining γ*'s* are from poles of types (1)–(3).

In both types (a) and (b), the γ*'s* can actually represent poles, but neither in (c) nor (d) can the γ*'s* represent poles, and this is because in (c), the term

$$\prod_{1\leqslant i<j\leqslant m}[az_iz_jq^{y_i+y_j}]_\infty = \prod_{1\leqslant k<j\leqslant m}\left[aa^{-1}\gamma_k^{-1}\gamma_jq^{y_k-y_j}\right]_\infty$$
$$= \prod_{1\leqslant k<j\leqslant m}\left[\gamma_k^{-1}\gamma_jq^{y_k-y_j}\right]_\infty,$$

which will be canceled by the zeros in the term $\prod_{i\neq j;\, i,j=1}^{m}[z_iz_j^{-1}q^{y_i-y_j}]_\infty$ in the numerator, while in (d) $z_iz_j = a^{-1}q^{-y_i-y_j}$ so that

$$z_1z_2\dots z_m = \prod(ac_k)\prod\left(d_l^{-1}\right)\prod\left(c_s^{-1}\right)\prod(ad_s) \neq 1,$$

which is a contradiction.

Then we will have the following types of poles:

$$z_i = \begin{cases} (ac_j)^{-1}q^{-y_1}, & 1 \leqslant j \leqslant 3 \text{ and } 1 \leqslant i \leqslant m-1,\\ d_jq^{-y_i}, & 1 \leqslant j \leqslant m \text{ and } 1 \leqslant i \leqslant m-1,\\ c_jq^{-y_i}, & 1 \leqslant j \leqslant 3 \text{ and } 1 \leqslant i \leqslant m-1,\\ a^{-1}d_j^{-1}q^{-y_i}, & 1 \leqslant j \leqslant m \text{ and } 1 \leqslant i \leqslant m-1, \end{cases} \tag{2.10}$$

while z_m can be expressed as follows:

$$z_m = (\gamma_1\gamma_2\dots\gamma_{m-1})_{-1},$$

which can be written as

$$z_m = \prod(ac_jq^{y_1})\prod\left(d_j^{-1}q^{y_i}\right)\prod\left(c_j^{-1}q^{y_i}\right)\prod(ad_jq^{y_i}) \quad \text{for some } i, j. \tag{2.11}$$

Now we are looking for those terms in the series $F_\delta(m-1)$ which contain the factor $1 - q^kd_1\dots d_m$, in their denominator. Considering the factors of types (1)–(3) listed above, we see that neither the factors $\prod_{i=1}^{m}\prod_{j=1}^{3}[ac_j\gamma_iq^{y_i}]_\infty$ or the factors $\prod_{1\leqslant i<j\leqslant m}[a\gamma_i\gamma_jq^{y_i+y_j}]_\infty$ will be divisible by $1 - q^kd_1\dots d_m$. Hence the terms of the series $F_\delta(m-1)$ which may not vanish after multiplying by $[d_1\dots d_m]_\infty$, and setting $d_1\dots d_m = q^{-k}$ are those for which, for some permutation $\rho \in S_m$, $\gamma_1 = d_{\rho(1)}$, $\gamma_2 = d_{\rho(2)}$, $\dots$, $\gamma_{m-1} = d_{\rho(m-1)}$ and $\gamma_m = d_{\rho(1)}^{-1}d_{\rho(2)}^{-1}\dots d_{\rho(m-1)}^{-1} = q^kd_{\rho(m)}$. The factor in the denominator that will be divisible by $1 - q^kd_1\dots d_m$ is $[d_m\gamma_m^{-1}q^{-y_j}]_\infty$, that is $[d_md_1d_2\dots d_{m-1}q^{-y_j}]_\infty = [d_1d_2\dots d_{m-1}d_mq^{-y_j}]_\infty$. With the permutation $\rho \in S_m$ as above, if we set $y_i = x_{\rho(i)}$ for $1 \leqslant i \leqslant m-1$ and $y_m = x_{\rho(m)}q^{-k}$, then for all j, $1 \leqslant j \leqslant m$, we will have $\gamma_jq^{y_j} = d_{\rho(j)}q^{x_{\rho(j)}}$ with $\sum_{j=1}^{m}x_j = k$.

In the following section we will give the prove of the even case of (1.2).

2.3. Even case

Let us start our proof by expanding the integral

$$\frac{2^n n! \prod_{1\leqslant i<j\leqslant 3}[a^{n+1}c_ic_j]_\infty}{(2\pi i)^{2n-1}[q]_\infty^n \prod_{1\leqslant i<j\leqslant 3}[ac_ic_j]_\infty} \int_{T^{2n-1}} \frac{\prod_{i=1}^{2n}\Big[a^{n+1}\prod_{k=1}^{3}c_k z_i^{-1}\Big]_\infty \prod_{i\neq j;\, i,j=1}^{2n}[z_iz_j^{-1}]_\infty}{\prod_{i=1}^{2n}\prod_{j=1}^{3}[ac_jz_i]_\infty \prod_{1\leqslant i<j\leqslant 2n}[az_iz_j]_\infty}$$
$$\times \frac{\prod_{j=1}^{2n}\Big[a^n\prod_{i=1}^{2n+1}d_iz_j^{-1}\Big]_\infty}{\prod_{i=1}^{2n+1}\prod_{j=1}^{2n}[d_iz_j^{-1}]_\infty}\frac{dz_1}{z_1}\cdots\frac{dz_{2n-1}}{z_{2n-1}},$$

which is integral (2.4) of [1], as a sum of residues, and rewrite it in the form

$$\frac{2^n n! \prod_{1\leqslant i<j\leqslant 3}[a^{n+1}c_ic_j]_\infty}{(2\pi i)^{2n-1}[q]_\infty^n \prod_{1\leqslant i<j\leqslant 3}[ac_ic_j]_\infty} \int_{T^{2n-1}} \frac{\prod_{i=1}^{2n}[d_{2n+1}z_i^{-1}]_\infty \prod_{i\neq j;\, i,j=1}^{2n}[z_iz_j^{-1}]_\infty}{\prod_{i=1}^{2n}\prod_{j=1}^{3}[ac_jz_i]_\infty \prod_{1\leqslant i<j\leqslant 2n}[az_iz_j]_\infty}$$
$$\times \frac{\prod_{j=1}^{2n}\Big[a^n\prod_{i=1}^{2n+1}d_iz_j^{-1}\Big]_\infty}{\prod_{i=1}^{2n}\prod_{j=1}^{2n}[d_iz_j^{-1}]_\infty \prod_{j=1}^{2n}[d_{2n+1}z_j^{-1}]_\infty}\frac{dz_1}{z_1}\cdots\frac{dz_{2n-1}}{z_{2n-1}}, \tag{2.12}$$

with $d_{2n+1} = a^{n+1}\prod_{k=1}^{3} c_k$. Integral (2.12) can be rewritten in the following form:

$$\frac{2^n n! \prod_{1\leqslant i<j\leqslant 3}[a^{n+1}c_ic_j]_\infty}{(2\pi i)^{2n-1}[q]_\infty^n \prod_{1\leqslant i<j\leqslant 3}[ac_ic_j]_\infty}$$
$$\times \int_{T^{2n-1}} \frac{\prod_{i\neq j;\, i,j=1}^{2n}[z_iz_j^{-1}]_\infty \prod_{j=1}^{2n}\Big[a^n\prod_{i=1}^{2n+1}d_iz_j^{-1}\Big]_\infty}{\prod_{i=1}^{2n}\prod_{j=1}^{3}[ac_jz_i]_\infty \prod_{1\leqslant i<j\leqslant 2n}[az_iz_j]_\infty \prod_{i,j=1}^{2n}[d_iz_j^{-1}]_\infty}\frac{dz_1}{z_1}\cdots\frac{dz_{2n-1}}{z_{2n-1}}. \tag{2.13}$$

Let each $z_i = \gamma_i q^{y_i}$, $i = 1,\ldots,2n$, and $y_i \in Z$. Then we can rewrite the integral (2.13) as

$$M\sum \frac{\prod_{i\neq j;\, i,j=1}^{2n}[\gamma_iq^{y_i}\gamma_j^{-1}q^{-y_j}]_\infty \prod_{j=1}^{2n}\Big[a^n\prod_{i=1}^{2n+1}d_i\gamma_j^{-1}q^{-y_j}\Big]_\infty}{\prod_{i=1}^{2n}\prod_{j=1}^{3}[ac_j\gamma_iq^{y_i}]_\infty \prod_{1\leqslant i<j\leqslant 2n}[a\gamma_i\gamma_jq^{y_i+y_j}]_\infty \prod_{i,j=1}^{2n}[d_i\gamma_j^{-1}q^{-y_j}]_\infty}, \tag{2.14}$$

where

$$M = \frac{2^n n! \prod_{1\leqslant i<j\leqslant 3}[a^{n+1}c_ic_j]_\infty}{[q]_\infty^n \prod_{1\leqslant i<j\leqslant 3}[ac_ic_j]_\infty}, \tag{2.15}$$

and the sum is taken under the residues of the poles of $z_i = \gamma_i q^{y_i}$, $i = 1,\ldots,2n-1$.

We have to prove that all the series are identical. First consider the special case of the algebra A_2 and let $n=2$ in the series (2.14). Then series (2.14) reduces to

$$M\sum_{i,j}\frac{\prod_{i\neq j;\,i,j=1}^{4}[\gamma_i q^{y_i}\gamma_j^{-1}q^{-y_j}]_\infty\prod_{j=1}^{4}\Big[a^2\prod_{i=1}^{5}d_i\gamma_j^{-1}q^{-y_j}\Big]_\infty}{\prod_{i=1}^{4}\prod_{j=1}^{3}[ac_j\gamma_i q^{y_i}]_\infty\prod_{1\leqslant i<j\leqslant 4}[a\gamma_i\gamma_j q^{y_i+y_j}]_\infty\prod_{i,j=1}^{4}[d_i\gamma_j^{-1}q^{-y_j}]_\infty}. \tag{2.16}$$

Now, we have $\prod_{i=1}^{4}\gamma_i=1$, that is $\gamma_4=\prod_{i=1}^{3}\gamma_i^{-1}$. By letting $d_1d_2d_3d_4=q^{-m}$, each $\gamma_i q^{y_i}$ by $d_{\sigma(i)}q^{x_{\sigma(i)}}$, $\gamma_j q^{y_j}$ by $d_{\sigma(j)}q^{x_{\sigma(j)}}$ and $\sum y_i=0$ for $i,j=1,2,3,4$ in series (2.16), then

$$M\sum_{i,j}\frac{\prod_{i\neq j;\,i,j=1}^{4}[d_{\sigma(i)}d_{\sigma(j)}^{-1}q^{x_{\sigma(i)}-x_{\sigma(j)}}]_\infty}{\prod_{i=1}^{4}\prod_{j=1}^{3}[ac_jd_{\sigma(i)}q^{x_{\sigma(i)}}]_\infty\prod_{1\leqslant i<j\leqslant 4}[ad_{\sigma(i)}d_{\sigma(j)}q^{x_{\sigma(i)}+x_{\sigma(j)}}]_\infty}\frac{\prod_{j=1}^{4}\Big[a^2\prod_{i=1}^{5}d_id_{\sigma(j)}^{-1}q^{-x_{\sigma(j)}}\Big]_\infty}{\prod_{i,j=1}^{4}[d_id_{\sigma(j)}^{-1}q^{-x_{\sigma(j)}}]_\infty}. \tag{2.17}$$

On replacing each x by y, all the series in (2.16) will be identical.

Now, to prove that all the series in (2.14) are identical, we will have $\prod_{i=1}^{2n}\gamma_i=1$, that is $\gamma_{2n}=\prod_{i=1}^{2n-1}\gamma_i^{-1}$, by letting $d_1d_2\ldots d_{2n}=q^{-m}$ and replacing each $\gamma_i q^{y_i}$ by $d_{\sigma(i)}q^{x_{\sigma(i)}}$, $\gamma_j q^{y_j}$ by $d_{\sigma(j)}q^{x_{\sigma(j)}}$, and $\sum y_i=0$ for $i,j=1,2,\ldots,2n$ in the series (2.14), then

$$M\sum_{i,j}\frac{\prod_{i\neq j;\,i,j=1}^{2n}[d_{\sigma(i)}d_{\sigma(j)}^{-1}q^{x_{\sigma(i)}-x_{\sigma(j)}}]_\infty}{\prod_{i=1}^{2n}\prod_{j=1}^{3}[ac_jd_{\sigma(i)}q^{x_{\sigma(i)}}]_\infty\prod_{1\leqslant i<j\leqslant 2n}[ad_{\sigma(i)}d_{\sigma(j)}q^{x_{\sigma(i)}+x_{\sigma(j)}}]_\infty}\frac{\prod_{j=1}^{2n}\Big[a^n\prod_{i=1}^{2n+1}d_id_{\sigma(j)}^{-1}q^{-x_{\sigma(j)}}\Big]_\infty}{\prod_{i,j=1}^{2n}[d_id_{\sigma(j)}^{-1}q^{-x_{\sigma(j)}}]_\infty}. \tag{2.18}$$

Substituting each x by y, there will be $(2n)!$ identical series in the series (2.14). That is

$$\begin{aligned}
&M(2n)!\sum_{y_1+\cdots+y_{2n}=0}\frac{\prod_{i\neq j;\,i,j=1}^{2n}[\gamma_i q^{y_i}\gamma_j^{-1}q^{-y_j}]_\infty\prod_{j=1}^{2n}\Big[a^n\prod_{i=1}^{2n+1}d_i\gamma_j^{-1}q^{-y_j}\Big]_\infty}{\prod_{i=1}^{2n}\prod_{j=1}^{3}[ac_j\gamma_i q^{y_i}]_\infty\prod_{1\leqslant i<j\leqslant 2n}[a\gamma_i\gamma_j q^{y_i+y_j}]_\infty\prod_{i,j=1}^{2n}[d_i\gamma_j^{-1}q^{-y_j}]_\infty}\\
&=\frac{(2n)!2^nn!\prod_{k=1}^{2n+1}\Big[a^{2n+1}d_k^{-1}\Big(\prod_{i=1}^{2n}d_i\Big)\prod_{k=1}^{3}c_k\Big]_\infty}{[q]_\infty^{3n-1}[a^n]_\infty\Big[a^{n+1}d_{2n+1}^{-1}\Big(\prod_{i=1}^{2n}d_i\Big)\prod_{k=1}^{3}c_k\Big]_\infty}\frac{\prod_{k=1}^{3}\Big[c_k^{-1}a^{n+1}\Big(\prod_{i=1}^{2n}d_i\Big)\prod_{k=1}^{3}c_k\Big]_\infty}{\prod_{1\leqslant i<j\leqslant 2n}[ad_id_j]_\infty\prod_{i=1}^{2n}\prod_{j=1}^{3}[ad_ic_j]_\infty\prod_{1\leqslant i<j\leqslant 3}[ac_ic_j]_\infty}.
\end{aligned} \tag{2.19}$$

Substituting for M from (2.15), series (2.19) takes the following form:

$$\begin{aligned}
&\sum_{y_1+\cdots+y_{2n}=0}\frac{\prod_{i\neq j;\,i,j=1}^{2n}[\gamma_i\gamma_j^{-1}q^{y_i-y_j}]_\infty\prod_{j=1}^{2n}\Big[a^n\prod_{i=1}^{2n+1}d_i\gamma_j^{-1}q^{-y_j}\Big]_\infty}{\prod_{i=1}^{2n}\prod_{j=1}^{3}[ac_j\gamma_i q^{y_i}]_\infty\prod_{1\leqslant i<j\leqslant 2n}[a\gamma_i\gamma_j q^{y_i+y_j}]_\infty\prod_{i,j=1}^{2n}[d_i\gamma_j^{-1}q^{-y_j}]_\infty}\\
&=\frac{\prod_{k=1}^{2n+1}\Big[a^{2n+1}d_k^{-1}\Big(\prod_{i=1}^{2n}d_i\Big)\prod_{k=1}^{3}c_k\Big]_\infty}{[q]_\infty^{2n-1}[a^n]_\infty\prod_{1\leqslant i<j\leqslant 3}[a^{n+1}c_ic_j]_\infty\Big[\prod_{i=1}^{2n}d_i\Big]_\infty}\frac{\prod_{k=1}^{3}\Big[c_k^{-1}a^{n+1}\Big(\prod_{i=1}^{2n}d_i\Big)\prod_{k=1}^{3}c_k\Big]_\infty}{\prod_{1\leqslant i<j\leqslant 2n}[ad_id_j]_\infty\prod_{i=1}^{2n}\prod_{j=1}^{3}[ad_ic_j]_\infty},
\end{aligned} \tag{2.20}$$

where $d_{2n+1} = a^{n+1} \prod_{l=1}^{3} c_l$.

Now, let

$$\{\gamma_1, \dots, \gamma_{2n}\} = \left\{d_1, \dots, d_{2n-1}, d_1^{-1} \dots d_{2n-1}^{-1}\right\} = \{d_1, \dots, d_{2n-1}, d_{2n} q^m\}. \tag{2.21}$$

Then

$$\begin{aligned} \{\gamma_1 q^{y_1}, \dots, \gamma_{2n} q^{y_{2n}}\} &= \{d_1 q^{x_1}, \dots, d_{2n-1} q^{x_{2n-1}}, d_{2n} q^{x_{2n}}\} \\ &= \{d_1, \dots, d_{2n-1}, d_{2n} q^m\}. \end{aligned} \tag{2.22}$$

Also let $q^{y_i} = q^{x_{\sigma(i)}}$, $i = 1, \dots, 2n-1$, $q^{y_{2n}} = q^{-m+x_{\sigma(2n)}}$, where $x_1 + \cdots + x_{2n} = m$, and $\gamma_i = d_{\sigma(i)}$ for $i = 1, \dots, 2n-1$ and also $\gamma_{2n} = \gamma_1^{-1} \dots \gamma_{2n-1}^{-1} = \prod_{i=1}^{2n-1} d_{\sigma(i)}^{-1} = d_{\sigma(2n)} q^m$. With these notation the l.h.s. of (2.20) becomes

$$\begin{aligned} &\sum_{y_1+\cdots+y_{2n}=0} \frac{\prod_{i\neq j;\, i,j=1}^{2n-1} [\gamma_i \gamma_j^{-1} q^{y_i - y_j}]_\infty \prod_{i=1}^{2n} [\gamma_i \gamma_{2n}^{-1} q^{y_i - y_{2n}}]_\infty}{\prod_{i=1}^{2n-1} \prod_{j=1}^{3} [a c_j \gamma_i q^{y_i}]_\infty [a c_j \gamma_{2n} q^{y_{2n}}]_\infty \prod_{1\leqslant i<j\leqslant 2n-1} [a \gamma_i \gamma_j q^{y_i + y_j}]_\infty} \\ &\quad \times \frac{\prod_{j=1}^{2n-1} \Big[a^n \prod_{i=1}^{2n+1} d_i \gamma_j^{-1} q^{-y_j}\Big]_\infty \Big[a^n \prod_{i=1}^{2n+1} d_i \gamma_{2n}^{-1} q^{-y_{2n}}\Big]_\infty}{\prod_{i,j=1}^{2n-1} [d_i \gamma_j^{-1} q^{-y_j}]_\infty [d_i \gamma_{2n}^{-1} q^{-y_{2n}}]_\infty \prod_{i=1}^{2n-1} [a \gamma_i \gamma_{2n} q^{y_i + y_{2n}}]_\infty} \end{aligned} \tag{2.23}$$

or equivalently,

$$\begin{aligned} &\sum_{x_1+\cdots+x_{2n}=m} \frac{\prod_{i\neq j;\, i,j=1}^{2n-1} [d_{\sigma(i)} d_{\sigma(j)}^{-1} q^{x_{\sigma(i)} - x_{\sigma(j)}}]_\infty \prod_{i=1}^{2n} [d_{\sigma(i)} d_{\sigma(2n)}^{-1} q^{x_{\sigma(i)} - x_{\sigma(2n)}}]_\infty}{\prod_{i=1}^{2n-1} \prod_{j=1}^{3} [a c_j d_{\sigma(i)} q^{x_{\sigma(i)}}]_\infty [a c_j d_{\sigma(2n)} q^{x_{\sigma(2n)}}]_\infty} \\ &\quad \times \frac{\prod_{j=1}^{2n-1} \Big[a^n \prod_{i=1}^{2n+1} d_i d_{\sigma(j)}^{-1} q^{-x_{\sigma(j)}}\Big]_\infty \Big[a^n \prod_{i=1}^{2n+1} d_i d_{\sigma(2n)}^{-1} q^{-x_{\sigma(2n)}}\Big]_\infty}{\prod_{1\leqslant i<j\leqslant 2n-1} [a d_{\sigma(i)} d_{\sigma(j)} q^{x_{\sigma(i)} + x_{\sigma(j)}}]_\infty \prod_{i=1}^{2n-1} [a d_{\sigma(i)} d_{\sigma(2n)} q^{x_{\sigma(i)} + x_{\sigma(2n)}}]_\infty} \\ &\quad \times \frac{1}{\prod_{i,j=1}^{2n-1} [d_i d_{\sigma(j)}^{-1} q^{-x_{\sigma(j)}}]_\infty [d_i d_{\sigma(2n)}^{-1} q^{-x_{\sigma(2n)}}]_\infty}. \end{aligned} \tag{2.24}$$

That is

$$\sum_{x_1+\cdots+x_{2n}=m} \frac{\prod_{i\neq j;\, i,j=1}^{2n} [d_i d_j^{-1} q^{x_i - x_j}]_\infty \prod_{j=1}^{2n} \Big[a^{2n+1} \Big(\prod_{i=1}^{2n} d_i\Big) \prod_{k=1}^{3} c_k d_j^{-1} q^{-x_j}\Big]_\infty}{\prod_{i=1}^{2n} \prod_{j=1}^{3} [a c_j d_i q^{x_i}]_\infty \prod_{1\leqslant i<j\leqslant 2n} [a d_i d_j q^{x_i + x_j}]_\infty \prod_{i,j=1}^{2n} [d_i d_j^{-1} q^{-x_j}]_\infty}. \tag{2.25}$$

Now, using the identity (1.3), series (2.25) can be rewritten as

$$\frac{\prod_{i\neq j;\, i,j=1}^{2n} [d_i d_j^{-1}]_\infty \prod_{j=1}^{2n} \Big[a^{2n+1} \Big(\prod_{i=1}^{2n} d_i\Big) \prod_{k=1}^{3} c_k d_j^{-1}\Big]_\infty}{\prod_{i=1}^{2n} \prod_{j=1}^{3} [a c_j d_i]_\infty \prod_{1\leqslant i<j\leqslant 2n} [a d_i d_j]_\infty \prod_{i,j=1}^{2n} [d_i d_j^{-1}]_\infty}$$

$$\times \sum_{x_1+\cdots+x_{2n}=m} \frac{\prod_{i=1}^{2n}\prod_{j=1}^{3}[ac_jd_i]_{x_i} \prod_{1\leqslant i<j\leqslant 2n}[ad_id_j]_{x_i+x_j} \prod_{i,j=1}^{2n}[d_id_j^{-1}]_{-x_j}}{\prod_{i\neq j;\, i,j=1}^{2n}[d_id_j^{-1}]_{x_i-x_j} \prod_{j=1}^{2n}\Big[a^{2n+1}\Big(\prod_{i=1}^{2n}d_i\Big)\prod_{k=1}^{3}c_kd_j^{-1}\Big]_{-x_j}} \tag{2.26}$$

upon which, we will have

$$\begin{aligned}
&\frac{\prod_{i\neq j;\, i,j=1}^{2n}[d_id_j^{-1}]_\infty \prod_{j=1}^{2n}\Big[a^{2n+1}\Big(\prod_{i=1}^{2n}d_i\Big)\prod_{k=1}^{3}c_kd_j^{-1}\Big]_\infty}{\prod_{i=1}^{2n}\prod_{j=1}^{3}[ac_jd_i]_\infty \prod_{1\leqslant i<j\leqslant 2n}[ad_id_j]_\infty \prod_{i,j=1}^{2n}[d_id_j^{-1}]_\infty} \\
&\quad\times \sum_{x_1+\cdots+x_{2n}=m} \frac{\prod_{i=1}^{2n}\prod_{j=1}^{3}[ac_jd_i]_{x_i} \prod_{1\leqslant i<j\leqslant 2n}[ad_id_j]_{x_i+x_j} \prod_{i,j=1}^{2n}[d_id_j^{-1}]_{-x_j}}{\prod_{i\neq j;\, i,j=1}^{2n}[d_id_j^{-1}]_{x_i-x_j} \prod_{j=1}^{2n}\Big[a^{2n+1}\Big(\prod_{i=1}^{2n}d_i\Big)\prod_{k=1}^{3}c_kd_j^{-1}\Big]_{-x_j}} \\
&= \frac{\prod_{k=1}^{2n+1}\Big[a^{2n+1}d_k^{-1}\Big(\prod_{i=1}^{2n}d_i\Big)\prod_{k=1}^{3}c_k\Big]_\infty}{[q]_\infty^{2n-1}[a^n]_\infty \prod_{1\leqslant i<j\leqslant 3}[a^{n+1}c_ic_j]_\infty} \frac{\prod_{k=1}^{3}\Big[c_k^{-1}a^{n+1}\Big(\prod_{i=1}^{2n}d_i\Big)\prod_{k=1}^{3}c_k\Big]_\infty}{\Big[\prod_{i=1}^{2n}d_i\Big]_\infty \prod_{1\leqslant i<j\leqslant 2n}[ad_id_j]_\infty \prod_{i=1}^{2n}\prod_{j=1}^{3}[ad_ic_j]_\infty},
\end{aligned} \tag{2.27}$$

that is

$$\begin{aligned}
&\sum_{x_1+\cdots+x_{2n}=m} \frac{\prod_{i=1}^{2n}\prod_{j=1}^{3}[ac_jd_i]_{x_i} \prod_{1\leqslant i<j\leqslant 2n}[ad_id_j]_{x_i+x_j} \prod_{i,j=1}^{2n}[d_id_j^{-1}]_{-x_j}}{\prod_{i\neq j;\, i,j=1}^{2n}[d_id_j^{-1}]_{x_i-x_j} \prod_{j=1}^{2n}\Big[a^{2n+1}\Big(\prod_{i=1}^{2n}d_i\Big)\prod_{k=1}^{3}c_kd_j^{-1}\Big]_{-x_j}} \\
&= \frac{[d_{2n}d_{2n}^{-1}]_\infty[a^nq^{-m}]_\infty \prod_{1\leqslant i<j\leqslant 3}[a^{n+1}c_ic_jq^{-m}]_\infty}{[a^n]_\infty \prod_{1\leqslant i<j\leqslant 3}[a^{n+1}c_ic_j]_\infty[q^{-m}]_\infty}.
\end{aligned} \tag{2.28}$$

Now, using the identity (1.4), we can rewrite the r.h.s. of (2.28) as follows:

$$\frac{[d_{2n}d_{2n}^{-1}]_\infty}{[q^{-m}]_\infty}[a^nq^{-m}]_m \prod_{1\leqslant i<j\leqslant 3}[a^{n+1}c_ic_jq^{-m}]_m.$$

Consequently, we will have

$$\begin{aligned}
&\sum_{x_1+\cdots+x_{2n}=m} \frac{\prod_{i=1}^{2n}\prod_{j=1}^{3}[ac_jd_i]_{x_i} \prod_{1\leqslant i<j\leqslant 2n}[ad_id_j]_{x_i+x_j} \prod_{i,j=1}^{2n}[d_id_j^{-1}]_{-x_j}}{\prod_{i\neq j;\, i,j=1}^{2n}[d_id_j^{-1}]_{x_i-x_j} \prod_{j=1}^{2n}\Big[a^{2n+1}\Big(\prod_{i=1}^{2n}d_i\Big)\prod_{k=1}^{3}c_kd_j^{-1}\Big]_{-x_j}} \\
&= \frac{[a^nq^{-m}]_m \prod_{1\leqslant i<j\leqslant 3}[a^{n+1}c_ic_jq^{-m}]_m}{[q^{-m}]_m},
\end{aligned} \tag{2.29}$$

where

$$\frac{[d_{2n}d_{2n}^{-1}]_\infty}{[q^{-m}]_\infty} = \frac{1}{[q^{-m}]_m}.$$

On using the identity (1.5), series (2.29) can be rewritten as follows:

$$\sum_{x_1+\cdots+x_{2n}=m} \frac{\prod_{i=1}^{2n}\prod_{j=1}^{3}[ac_jd_i]_{x_i} \prod_{1\leqslant i<j\leqslant 2n}[ad_id_j]_{x_i+x_j} \prod_{i,j=1}^{2n}[d_id_j^{-1}]_{-x_j}}{\prod_{i\neq j;\, i,j=1}^{2n}[d_id_j^{-1}]_{x_i-x_j} \prod_{j=1}^{2n}\Big[a^{2n+1}\Big(\prod_{i=1}^{2n}d_i\Big)\prod_{k=1}^{3}c_kd_j^{-1}\Big]_{-x_j}}$$
$$= \frac{[1]_{-m}}{[a^n]_{-m}\prod_{1\leqslant i<j\leqslant 3}[a^{n+1}c_ic_j]_{-m}}, \tag{2.30}$$

which completes the proof of the even case.

The proof of the odd case of Theorem 1.2 is entirely similar to the even case. The only difference is that one starts with a different r.h.s. from identity (1.1). □

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